Bounds on the Optimal Elasticity Parameters for a Snake *

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Abstract. This paper develops a formalism by which an estimate for the upper and lower bounds for the elasticity parameters for a snake can be obtained. Objects different in size and shape give rise to different bounds. The bounds can be obtained based on an analysis of the shape of the object of interest. Experiments on synthetic images show a good correlation between the estimated behaviour of the snake and the actually observed. Experiments on real X-ray images show that the parameters for optimal segmentation lie within the estimated bounds.

Keywords: snakes, elasticity parameters, segmentation.

1 Introduction

Snakes, introduced by Kass et al.[2], have several advantages compared to traditional segmentation techniques specially in case of noisy images and partly occluded objects [1, 2]. They are a model-based technique since ways exist to incorporate information about the object of interest into the snake formalism. An initial snake can be defined specifying it in accordance with the object of interest. Besides, it is possible by controlling the internal energy to constrain the smoothness of the snake. The smoothness required for an accurate segmentation is controlled by the elasticity parameters and is closely related to the shape of the object.

Despite the great interest in snakes, estimation of the elasticity parameters have not been given much attention previously. Samadani [5] dynamically estimates and adjusts the parameters to avoid instability in the deformation proces. In contrast with his work, our purpose is to establish some bounds of the optimal parameters of elasticity that provide the necessary smoothness of the snake. We see as a natural way to do it by extracting information from the object model.

We present here a new formalism by which the upper and lower bounds of the elasticity parameters for a snake can be obtained through an analysis of the shape of the model. First the theoretical foundation for the existence of bounds on the parameters is developed independently of the chosen snake implementation. Based on an implementation using the Finite Difference Method (FDM) formulas

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are then derived for calculating the bounds based on the shape of the object. At last experimental results are presented to determine the correlation between the estimated behaviour of the snake and the actually observed.

2 Parameters of elasticity

A snake is an elastic curve $u(s) = (x(s), y(s))$ for which an energy function E_{snake} is defined based on an internal energy E_{int} and an external energy E_{ext},

$$E_{snake} = \int_0^1 E_{int}(u(s)) + E_{ext}(u(s))\, ds.$$

The internal energy is given by the sum of the membrane energy and the thin-plate energy: $E_{int}(u) = \alpha E_{membrane} + \beta E_{thin-plate} = \alpha u'(s) + \beta u''(s)$. The parameters α and β are the parameters of elasticity. α controls the stretching and β the bending of the snake curve.

The external energy is obtained from a potential field derived as the image gradient [2] or as a distance map of the edge points [1]. Without loss of generality we use the latter definition leading to the following expression, $E_{ext}(u(s)) = P(x, y) = d(x, y)$, where $d(x, y)$ is the distance between pixel (x, y) and the closest edge point. Minimizing the external energy the snake is attracted towards the edge points of the image. Minimizing the internal energy the snake shape is smoothed. The result of the segmentation is obtained when the snake detects a minimum of the total energy.

High values of the parameters of elasticity put great weight on the internal energies and consequently give a smooth curve. In general we want the parameters to be as high as possible. It will decrease the risk of the snake being attracted to spurious noisy edges during its iterations [1]. However, if the parameters become too high, problems in terms of contour surpassing take place.

Let us consider a snake, u_c, of length n placed in the potential valley corresponding to the contour of the object we are looking for. And let ϵ be some predetermined constant. The following definitions can be stated,

Definition 1 *The minimum area around the contour valley containing all possible deformed versions, u_t, of snake u_c in a distance less than ϵ to u_c ($||u_t - u_c|| < \epsilon n$) is called the* **$\epsilon$-neighbourhood** *of the contour valley.*

Definition 2 **Local surpassing** *is when the snake u_c in the process of energy minimization moves locally out of the contour valley.* **Global surpassing** *is when the snake u_c deforms to a snake outside an ϵ-neighbourhood of the contour valley given a constant ϵ.*

An illustration of local and global surpassing is given in figure 1.

Definition 3 *The maximum values of the elasticity parameters for a snake that do not give rise to any local surpassing are called* **local (surpassing) parameters***. The maximum values that given some ϵ do not give rise to any global surpassing are called* **global (surpassing) parameters***.*

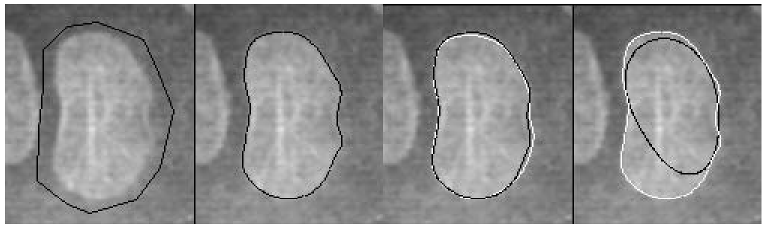

Fig. 1. Three different situations for segmentation of a bone structure in a X-ray image; from left to the right: an initial snake, the snake shrunk to the contour, a local surpassing of the contour shown in white by the snake shown in black and a global surpassing by the snake.

The fact that the local parameters are the maximum parameters that do not give rise to any local surpassing from the contour means that the deforming snake by these parameters of elasticity will not get shapes more stretched and bended than this of the object model. Due to a monotony property of the parameters of elasticity each pair of parameters greater than the local parameters will not allow discontinuities in the snake not proper to the object model. Hence, the local parameters can be thought of as lower bounds of the parameters giving an optimal segmentation. The global parameters are the maximum values for which the displacement of the snake does not exceed some ϵ-neighbourhood of the contour. It is to be determined by the user as the area around the contour where the snake does not lose its influence. The point for global parameters therefore determine the upper bounds of the parameters of elasticity.

Smoothing its shape the snake has the tendency to shrink itself. Based on the properties of the global parameters of elasticity it can be shown that the initial snake placed around the object is able to fall into an ϵ-neighbourhood of the object contour. This paper therefore proposes an approach to determine the bounds by an analysis of the energy when the snake surpasses the contour.

2.1 Fundamentals

Before being able to determine some values for the parameters bounds it is necessary to see some fundamental relations concerning certain properties of the parameters of elasticity. The fundamentals are stated without proofs, for more details see [3]. It should be emphasized that relations described in this section do not depend on any specific snake implementation, neither on any specific calculation of the potential field.

Let us consider the initial snake u_0, the snake u_c shrunk to the contour valley and the snake u_t obtained from the snake curve u_c. Let $E^c_{membrane}$, $E^c_{thin-plate}$, P^c, $E^t_{membrane}$, $E^t_{thin-plate}$ and P^t are the membrane energy, the thin-plate energy and the potential energy of the snakes u_c and u_t, respectively.

Lemma 1. *Let the parameters of elasticity* (α', β') *be such that the snake leaves* *the ϵ-neighbourhood of the contour valley given some ϵ. Then for any greater parameters of elasticity $\alpha \geq \alpha'$ and $\beta \geq \beta'$ the snake leaves the ϵ-neighbourhood.*

Lemma 2. *Let the parameters of elasticity (α', β') be such that the snake remains in the ϵ-neighbourhood of the contour valley given some ϵ. Then for any smaller parameters of elasticity $\alpha \leq \alpha'$ and $\beta \leq \beta'$ the snake remains in the ϵ-neighbourhood.*

These lemmas define a kind of monotony of the parameters of elasticity. They show that for a given parameters setting found to give local or global surpassing any parameters values greater than the ones found will also cause surpassing.

Theorem 3. *The maximum pair of parameters of elasticity that retains the snake u_c in an ϵ-neighbourhood of the contour is given by the formulas,*

$$\alpha = \frac{P^t - P^c}{2(E^c_{membrane} - E^t_{membrane})}, \quad \beta = \frac{P^t - P^c}{2(E^c_{thin-plate} - E^t_{thin-plate})} \tag{1}$$

where the snake u_t is obtained from the snake u_c and $||u_t - u_c|| = \epsilon n$.

2.2 Calculating the bounds of the optimal parameters of elasticity

Given a specific shape model, the task now becomes one of determining the parameters not giving rise to local and global surpassing with respect to some constant ϵ. They can be obtained from formulas (1) substituting the estimations of the energies of the snake u_c placed in the contour valley and its deformation u_t in a distance ϵ from the contour. Without loss of generality we compute the parameters for the snake implementation by FDM and use the fact that the potential is generated as a distance map of the edge points of the original image. In order to simplify our calculus we ensure that the snake pixels are evenly distributed on the curve in each iteration of the snake movement.

The energies of u_c can be estimated by the initial snake u_0 and the contour model [3]. We obtain the global parameters substituting the energies of the snake u_t after a global surpassing in a distance ϵ. Thus, we get,

$$\alpha_{gl} = \frac{kn^2}{8\pi(n - \epsilon k\pi)}, \quad \beta_{gl} = \frac{kn^4}{32\pi^3(n - \epsilon k\pi)} \tag{2}$$

where $k = \frac{u_0}{u_c}$ is the accumulation rate of the initial snake in the contour valley.

Corollary 4. *Between the global parameters the following relation is valid,*

$$\beta_{gl} = \frac{n^2}{4\pi^2}\alpha_{gl}.$$

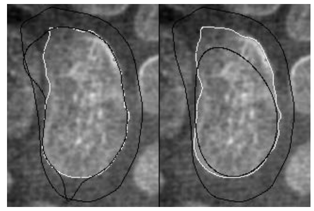

Fig. 2. Average pixel distance between the contour and a snake deformed for different settings near to the global parameters (α, β).

For the case of the local parameters we use the fact that local surpassing firstly occurs in the most curved place of the snake. A deformed snake after a local surpassing there is used. From formulas (1) for the local parameters we get,

$$\alpha_{loc} = \frac{\theta\lceil\frac{m}{2}\rceil k^2}{2m(1 - 2k^2(r+\epsilon)^2(1-\cos\frac{\mu}{m}))}, \quad \beta_{loc} = \frac{\theta\lceil\frac{m}{2}\rceil r^2}{2m(1 - 4r^2(r+\epsilon)^2(1-\cos\frac{\mu}{m})^2)}$$

$$\theta = \sqrt{r^2 + (r+\epsilon)^2 - 2r(r+\epsilon)\cos\frac{\nu-\mu}{2}} - \epsilon, \; \mu = \arccos\left(\frac{\epsilon(2r+\epsilon) + r^2\cos(m \arccos(1-\frac{1}{2r^2}))}{(r+\epsilon)^2}\right)$$

where m, ν and r are the length, the angle and the radius of the more curved segment of the snake u_c.

3 Results and discussion

Two lines of experiments have been conducted. One based on synthetic images with the purpose of analyzing the precision of the estimated effect of the parameters in relation to the actually observed effect. Another line includes real X-ray images of bone structures with the purpose of showing the usefulness of the local and global parameters in terms of lower and upper bounds of the optimal parameters for model-based segmentation.

Six different types of objects have been used for the experiments: circle, octagon, hexagon, square, rectangle and triangle. For each object the values of α and β from formulas (1) were computed given $\epsilon = 1.5$. Firstly, 5 scaled versions of each object was produced having length 100, 200... 500 pixels. Computing the displacement of the snake from the contour after the snake stabilizing, the experiments shew no dependence of the precision on the snake length.

Secondly, an experiment was conducted in which different settings of the parameters α and β around the calculated setting were used. It can clearly be seen in figure 2 that the difference between the initial snake and the deformed snake is a monotone increasing function as predicted by Lemma 1 and 2. The graphs show that the majority of the objects lie close to the desired distance $\epsilon = 1.5$ with a non significant tendency to exceed the threshold. Objects with sharp corners, e.g. the triangle give rise to less precise estimates. The test also shows that the distance function does not increase rapidly in the interval closely

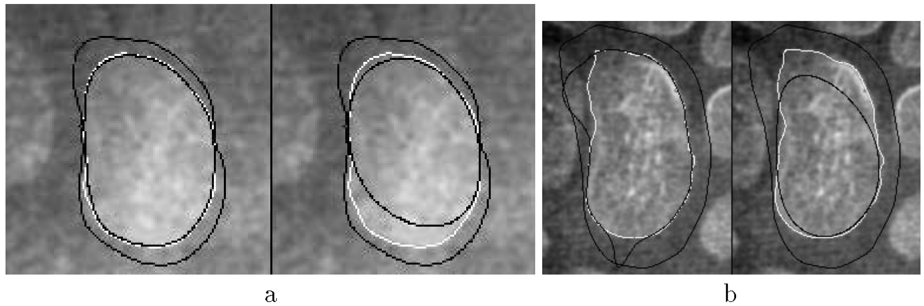

Fig. 3. Two different situations of using local (left) and global (right) parameters

surrounding the global values. This clearly indicates that the calculated global values should be taken more as an indication of an interval in which similar effects can be achieved.

In the test on real images of hand radiographs for each bone we used as an initial snake a model with similar structure but differing in size. Depending on the difference between the initial snake and the object contour in some cases the snake slided correctly to the contour (Figure 3a (left)). The snake was retained outside of the object contour in cases of more different initial snake and contour due to attraction by near edge points (Figure 3b (left)).

In accordance to our observations the global parameters assure us that the snake will surpass the contour going out in some distance with predetermined magnitude (Figure 3a (right) and Figure 3b (right)). In this way any snake detention in a local energy minimum before arriving to the contour will be avoided. Due to the monotony property of the elasticity parameters we can state that the parameters of the optimal segmentation belong to the intervals determined by the local and global parameters.

4 Conclusion

Based on the concept of local and global surpassing we have defined some local and global surpassing parameters which serve as lower and upper bounds on the elasticity parameters of a snake in terms of segmentation of a given object. Formulas are presented by which an estimate of the parameters bounds can be calculated for a given object model. Experiments show a good correlation between the estimate and results obtained on synthetic images. Experiments on real X-ray images show that optimal segmentation is only possible by parameters within the bounds defined in this paper. By establishing bounds on the parameters the search space for the values giving the best segmentation is limited. In [4] we propose guidelines for non-snake experts to determine the optimal parameters of elasticity based on an automatic inspection of the objects models.

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